Computation of the Quantum Rate-Distortion Function

K. He[1] J. Saunderson[1] H. Fawzi[2]

[1]Department of Electrical and Computer System Engineering
Monash University

[2]Department of Applied Mathematics and Theoretical Physics
University of Cambridge

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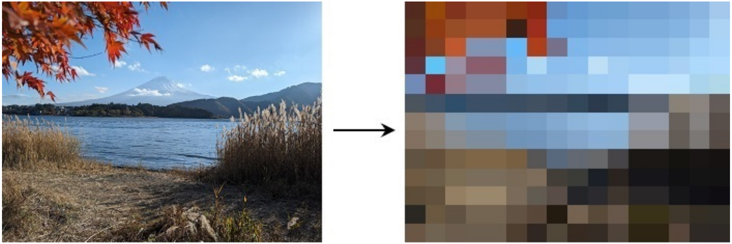
Introduction

Quantum rate-distortion problem: Consider the problem data

- Quantum signal $\rho \in \mathbb{H}^n_+$
- Distortion matrix $\Delta \in \mathbb{H}^{n^2}_+$
- Maximum allowable distortion $D \geq 0$



Input signal — Compressed signal

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Maximum achievable compression of quantum signal is

$$\begin{aligned} \min_{\sigma \in \mathbb{H}^{n^2}_+} \quad & \operatorname{tr}[\sigma \log(\sigma)] - \operatorname{tr}[\mathcal{A}(\sigma) \log(\mathcal{A}(\sigma))] \\ \text{subj. to} \quad & \operatorname{tr}[\sigma \Delta] \leq D, \quad \mathcal{B}(\sigma) = \rho, \end{aligned}$$

for linear operators $\mathcal{A} : \mathbb{H}^{n^2} \to \mathbb{H}^n$ and $\mathcal{B} : \mathbb{H}^{n^2} \to \mathbb{H}^n$

Quantum entropy: If λ_i are the eigenvalues of σ, then

$$\operatorname{tr}[\sigma \log(\sigma)] = \sum_{i=1}^{n^2} \lambda_i \log(\lambda_i)$$

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Problem dimension can scale quickly

- Need to optimize over $\mathbb{H}^{n^2}$, i.e., optimization problem scales $O(n^4)$
- Is there structure in problem we can take advantage of?

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Not many good algorithms available to compute quantum functions

- Approximate logarithms as linear matrix inequalities [Fawzi & Saunderson, 2023]
- Primal-dual interior point methods for general non-symmetric conic programs [Dahl & Andersen, 2022], [Coey et al., 2022], [Papp & Yildiz, 2022], [Karimi & Tuncel, 2020]

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- Use **mirror descent**

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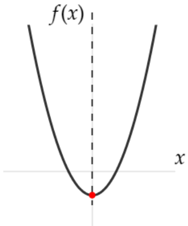
Preliminaries

Many optimization problems possess symmetries



If problem is convex, symmetries inform us about problem solutions

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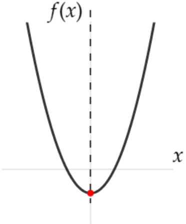

If problem is convex, symmetries inform us about problem solutions

A **representation** of a group $\mathcal{G}$ is a pair $(\mathbb{V}, \pi)$, where

- $\mathbb{V}$ is a vector space
- $\pi : \mathcal{G} \rightarrow GL(\mathbb{V})$ is a group homomorphism

e.g., for group $\mathcal{G}$ consisting of matrices, let $(\mathbb{H}^n, \pi)$ be a congruence trans.

$$\pi(g)(X) = gXg^\dagger, \quad \forall g \in \mathcal{G}$$

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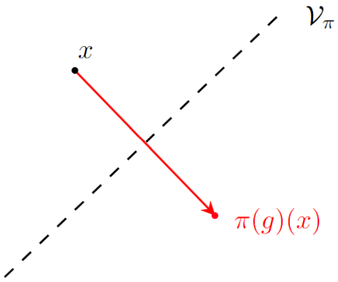

Fixed-point subspace $\mathcal{V}_\pi$: Set of all pnts. fixed under $\pi(g)$ for all $g \in \mathcal{G}$.

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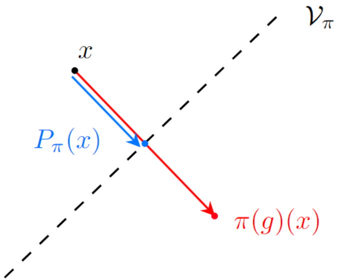

Fixed-point subspace $\mathcal{V}_\pi$: Set of all pnts. fixed under $\pi(g)$ for all $g \in \mathcal{G}$.

Projection operator P_π: Linear proj. onto the fixed-point subspace $\mathcal{V}_\pi$. Given by the group average

$$P_\pi = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \pi(g).$$

Lemma 1

Consider a representation $(\mathbb{V}, \pi)$ of a group $\mathcal{G}$. If a convex optimization problem

$$\min_{x} f(x), \quad \text{subj. to } x \in \mathcal{X},$$

is invariant under π, meaning

$$\begin{aligned} & f(\pi(g)(x)) = f(x) && \forall g \in \mathcal{G},\ \forall x \in \mathcal{X} \\ \text{and} \quad & \pi(g)(x) \in \mathcal{X} && \forall g \in \mathcal{G},\ \forall x \in \mathcal{X}, \end{aligned}$$

then there is an optimal point for the optimization problem in $\mathcal{V}_\pi$.

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Proof:

$$f^* \leq f\left(\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \pi(g)(x^*)\right) \leq \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} f(\pi(g)(x^*)) = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} f^* = f^*.$$

Quantum rate-distortion

Typically use the **entanglement fidelity** distortion matrix

$$\Delta = \sum_{ij}^{n} \sqrt{\lambda_i \lambda_j}\, v_i v_j^\dagger \otimes v_i v_j^\dagger, \quad \text{where} \quad \rho = \sum_{i=1}^{n} \lambda_i v_i v_i^\dagger.$$

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Theorem 2 (HSF, 2023)

Consider the group

$$\mathcal{G}_{ea} = \left\{ \sum_{i=1}^{n} z_i v_i v_i^\dagger : z \in \{\pm 1, \pm\sqrt{-1}\}^n \right\}$$

and corresponding representation $(\mathbb{H}^{n^2}, \pi_{cc})$ where

$$\pi_{cc}(g)(X) = (g \otimes \bar{g}) X (g \otimes \bar{g})^\dagger.$$

The quantum rate-distortion problem is invariant under this representation.

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Corollary 2 (HSF, 2023)

A solution to the quantum rate-distortion problem is in

$$\mathcal{V}_{ea} = \left\{ \sum_{i \neq j}^{n} \alpha_{ij} v_i v_i^\dagger \otimes v_j v_j^\dagger + \sum_{ij}^{n} \beta_{ij} v_i v_j^\dagger \otimes v_i v_j^\dagger : \alpha_{ij} \in \mathbb{R}\ \forall i \neq j,\ \beta \in \mathbb{H}^n \right\}.$$

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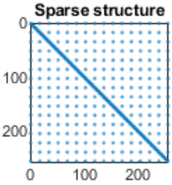
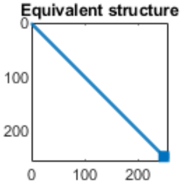
This subspace $\mathcal{V}_{ea} \subset \mathbb{H}^{n^2}$ has a real dimension of $2n^2 - n$

Quantum rate-distortion

Visualizing sparsity structure when v_i is the standard basis and $n = 16$:





Isomorphic to

- $n^2 - n$ blocks of size 1×1,
- ones block of size $n \times n$.

Easy to take eigendecomposition, quantum entropies, etc.

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Mirror descent

Consider constrained convex optimization problem

$$\min_{x \in \mathcal{X}} \quad f(x).$$

Projected gradient descent can be represented as

$$x^{k+1} = \arg\min_{x \in \mathcal{X}} \langle \nabla f(x^k), x \rangle + \frac{1}{2t_k} \|x - x^k\|_2^2$$

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Mirror descent replaces Euclidean norm with Bregman divergence

$$x^{k+1} = \arg\min_{x \in \mathcal{X}} \langle \nabla f(x^k), x \rangle + \frac{1}{t_k} D_\varphi(x \| y)$$

where

$$D_\varphi(x \| y) \coloneqq \varphi(x) - (\varphi(y) + \langle \nabla \varphi(y), x - y \rangle).$$

Mirror descent – convergence

A function f is L-smooth relative to φ if for $L > 0$

$$L\varphi - f \quad \text{convex}$$

Mirror descent w/ $t_k = 1/L$ converges sublinearly if f is L-smooth rel. to φ

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Theorem 3 (HSF, 2023)

The objective function of the quantum rate-distortion problem is 1-smooth relative to $\varphi(x) = \text{tr}[x \log(x)]$.

Therefore, mirror descent applied to QRD problem with unit step size and $\varphi(x) = \text{tr}[x \log(x)]$ will converge sublinearly to global optimum.

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Caveat:

- Each iteration requires solving a convex subproblem
- Can do efficiently by solving the dual problem inexactly (while retaining convergence guarantees!)

Numerical experiments

(a) Without symmetry reduction

			Ours		CvxQuad	
n	#variables	D	Time (s)	Gap	Time (s)	Gap
8	4×10^3	0.8	4.88	$2e{-}8$	Out of memory	
		0.5	1.12	$1e{-}8$	Out of memory	
32	1×10^6	0.8	> 3600.00	$2e{-}7$	Out of memory	
		0.6	1321.85	$1e{-}7$	Out of memory	
512	7×10^{10}	0.9	Out of memory		Out of memory	
		0.6	Out of memory		Out of memory	

(b) With symmetry reduction

			Ours		CvxQuad	
n	#variables	D	Time (s)	Gap	Time (s)	Gap
8	1×10^2	0.8	.17	$7e{-}9$	454.98	$2e{-}8$
		0.5	.07	$4e{-}9$	686.11	$6e{-}8$
32	2×10^3	0.8	1.52	$6e{-}8$	Out of memory	
		0.6	.32	$5e{-}9$	Out of memory	
512	5×10^5	0.9	2174.38	$7e{-}8$	Out of memory	
		0.6	1216.96	$5e{-}9$	Out of memory	

Conclusion

Summary:

- Rate-distortion problems possess symmetries that can be exploited to significantly reduce dimensionality of the optimization problem.
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Outlook:

- Other problems in quantum inf. theory w/ symmetries we can exploit?
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Outlook:

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Paper: `https://arxiv.org/abs/2309.15919`
Code: `https://github.com/kerry-he/efficient-qrd`